Game Theory Approach for the Integrated Design of Structures and Controls

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The problem of design of actively controlled structures subject to constraints on the damping parameters of the closed-loop system is formulated as a multiobjective optimization problem. The structural weight and the controlled system energy are considered as objective functions for minimization with cross-sectional areas of members as design variables. A computational procedure is developed for solving the multiobjective optimization problem using cooperative game theory. The feasibility of the procedure is demonstrated through the design of two truss structures.

Introduction

LARGE space structures face difficult problems of vibration control. Because they require low weight, such structures will lack the stiffness and damping necessary for the passive control of vibrations. Hence, current research is directed toward the design of active vibration-control systems for such structures. The objective of vibration control is to design the structure and its controls optimally to either eliminate vibration completely or to reduce the mean-square response of the system to a desired level within a reasonable span of time.

The structural designer wants to adjust the structural parameters to minimize the mass while achieving the desired frequencies, mode shapes, and dynamic response. The control designer, on the other hand, wishes to size the controllers (and possibly select the number and locations of the actuators and sensors) in order to minimize the energy of the vibrating structure.

A great deal of research is currently in progress on designing active vibration-control systems for large flexible space structures.[1] In Refs. 2 and 3, an optimal structure is initially designed to satisfy constraints on weight, strength, displacements, and frequency distribution, and then an optimum control system is designed to improve the dynamic response of the structure. In Refs. 4 and 5, a simultaneous integrated design of the structure and vibration-control system is achieved by improving the configuration as well as the control system. A unified approach to achieve satisfaction of eigenspace constraints is presented in Ref. 6.

The weight of the structure is minimized with constraints on distribution of the eigenvalues and/or damping parameters of the closed-loop system by Khot et al.,[7] Salama et al.,[5] and Miller and Shim[8] considered the simultaneous minimization, in structural and control variables, of the sum of structural weight and the infinite-horizon linear-regulator quadratic-control cost. The frequency control, the effect on the dynamic response of flexible structures, and the associated computational issues were discussed by Venkayya and Tischler[3] and Miller et al.[9] The structure/control system optimization problem was formulated by Khot et al.[10] with constraints on the closed-loop eigenvalue distribution and the minimum Frobenious norm of the control gains.

It can be seen that the combined structural/control optimization problem has not been formulated and solved as a multiobjective optimization problem. In the present work, the problem of design of actively controlled structure is formulated as a multiobjective or vector optimization problem. Usually there exists no unique solution that would give an optimum for all the objective functions simultaneously.[11,12] Hence a new optimality concept, different from that used in scalar optimization, is to be used in finding the solution of the vector optimization problem. The concept of Pareto-optimality has been found to be quite useful in this context. Several methods have been suggested in the literature for generating the Pareto-optimum solutions.[13-18] The simplest approach for dealing with multiple objectives involves the construction of a new objective function as a linear combination of the original objective functions. The approach requires the determination of relative weights for the various objective functions. This becomes an extremely difficult task when 1) the number of objective functions is large and 2) the units of different objectives are different. The game theory approach for generating the best compromise (Pareto-optimal) solution is presented in the present work.[19,20] This approach seeks to find an optimal compromise between the conflicting multiple objective functions. In addition, the method gives the relative weights of the various objective functions at optimum solution. The computational procedure is demonstrated through the design of two actively controlled truss structures.

Equations of Motion

The equations of motion of a large space structure with active controls under external forces are given by

$$[M]\ddot{U} + [C]\dot{U} + [K]U = [D] = [D]F \tag{1}$$

where $[M]$ is the mass matrix, $[C]$ the damping matrix, and $[K]$ the stiffness matrix. These matrices are of order r where r denotes the number of degrees of freedom of the structure; U represents the vector of displacements and a dot over a symbol denotes differentiation with respect to time, and $[D]$ is the $r \times p$ matrix denoting the applied load distribution that relates the control input vector F to the coordinate system. The number of components in F is assumed to be p. By introducing the coordinate transformation

$$U = [\phi]\eta \tag{2}$$

Received Nov. 25, 1986; revision received July 24, 1987.

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where $[\phi]$ is the $r\times r$ modal matrix whose columns are the eigenvectors and η is the vector of modal coordinates, Eq. (1) can be represented in state space form by assuming proportional damping as

$$\dot{x}=[A]x+[B]F \tag{3}$$

where

$$x=\left\{\begin{matrix}\eta\\ \dot{\eta}\end{matrix}\right\}$$

is the state variable vector of order $n\times 1$ where $n=2r$, $[A]$ is the $n\times n$ plant matrix, and $[B]$ the $n\times p$ input matrix given by

$$[A]=\begin{bmatrix}[O] & [I]\\ [-\omega^2] & [-2\zeta\omega]\end{bmatrix} \tag{4}$$

$$[B]=\begin{bmatrix}[O]\\ [\phi]^T[D]\end{bmatrix} \tag{5}$$

The symbol ζ is the vector of modal damping factors and ω the vector of natural frequencies of the structure.

In order to design a controller using a linear quadratic regulator, a performance index J is defined as

$$J=\int_0^\infty (x^T[Q]x+F^T[R]F)\mathrm{d}t \tag{6}$$

where $[Q]$ is the state weighting matrix that has to be positive semidefinite and $[R]$ the control weighting matrix that has to be positive definite. It is possible to control the damping response time, amplitudes of vibration, etc., of the system by proper selection of the elements of the matrices $[Q]$ and $[R]$. If $[Q]$ and $[R]$ are chosen as

$$[Q]=\begin{bmatrix}[K] & [O]\\ [O] & [M]\end{bmatrix} \tag{7}$$

and

$$[R]=[D]^T[K]^{-1}[D] \tag{8}$$

then Eq. (6) provides a measure of total system strain, kinetic, and potential energies.

The result of minimizing the quadratic performance index and satisfying the state equation gives the state feedback control law[13]

$$F=-[G]x \tag{9}$$

where $[G]$ is the optimum gain matrix given by

$$[G]=[R]^{-1}[B]^T[P] \tag{10}$$

with $[P]$ representing a symmetric positive definite matrix called the Riccati matrix, and is found by solving the following algebraic Riccati equation:

$$[A]^T[P]+[P][A]+[Q]-[P][B][R]^{-1}[B]^T[P]=[O] \tag{11}$$

Substituting Eq. (9) into Eq. (3) gives the governing equations for the optimal closed-loop system in the form

$$\dot{x}=[A_{cl}]x \tag{12}$$

where

$$[A_{cl}]=[A]-[B][G] \tag{13}$$

The eigenvalues of the closed-loop matrix $[A_{cl}]$ are a set of complex conjugate pairs written as

$$\lambda_i=\bar{\sigma}_i\pm j\bar{\omega}_i,\qquad i=1,2,\ldots,n \tag{14}$$

where $j=\sqrt{-1}$. The damping factors ξ_i and the damped frequencies $\bar{\omega}_i$ are related to the complex eigenvalues through

$$\xi_i=-\frac{\bar{\sigma}_i}{\sqrt{\bar{\sigma}_i^2+\bar{\omega}_i^2}} \tag{15}$$

Multiobjective Optimization Problem

The cross-sectional areas of the members of the structure are chosen as the design variables. The weight of the structure f_1 is given by

$$f_1(z)=\sum \rho_i A_i \ell_i \tag{16}$$

where ρ_i is the weight density, A_i the cross-sectional area, ℓ_i the length of element i, and $z=\{A_1A_2A_3\ldots\}^T$ the design vector. For any specified initial condition x_o it it well known that

$$\min_F \frac{1}{2}\int_0^\infty (x^T[Q]x+F^T[R]F)\,\mathrm{d}t=\frac{1}{2}x_o^T[P]x_o \tag{17}$$

and the corresponding optimal control is given by Eq. (9). The minimum value of the quadratic performance index is taken as the second objective function $f_2(z)$ as

$$f_2(z)=x_o^T[P]x_o \tag{18}$$

Other possible objective functions are the Frobenious norm and the effective damping response time. The third objective function $f_3(z)$ is taken as the Frobenious norm of the control gains (strictly speaking, square of the weighted norm) as[7]

$$f_3(z)=\mathrm{trace}\{[G]^T[R][G]\} \tag{19}$$

Essentially, the Frobenious norm of the control gains represents the value of the integrand of the quadratic control effort. By minimizing the value of the integrand one may hope to minimize the required control effort. The effective damping response time $f_4(z)$, under the action of an initial disturbance $x_o=x(t=0)$, can be expressed as[10]

$$f_4(z)=\frac{x_o^T[P]x_o}{x_o^T[Q]x_o} \tag{20}$$

The magnitude of f_4 indicates the effect of the control system in reducing vibrations.

Constraints are placed on the closed loop damping ratios as

$$\xi_i-\xi_i^{(o)}=0 \tag{21}$$

$$\xi_i-\xi_i^{(o)}\geq 0 \tag{22}$$

where $\xi_i^{(o)}$ is a specified value. Bounds are placed on the cross-sectional areas of members as

$$A_i^{(\ell)}\leq A_i\leq A_i^{(u)} \tag{23}$$

where the superscripts ℓ and u indicate lower and upper bound values.

Solution Procedure

The design problem stated in the previous section can be stated as a standard multiobjective nonlinear programming

problem as follows:
Find z that minimizes

$$f(z) = \begin{Bmatrix} f_1(z) \\ f_2(z) \\ \cdot \\ \cdot \\ \cdot \\ f_k(z) \end{Bmatrix} \tag{24}$$

subject to

$$g_i(z) \leq 0, \; j = 1,2,\ldots,m_1$$

and

$$h_j(z) = 0, \; j = 1,2,\ldots,m_2$$

The concept of the Pareto-optimal solution has been found to be useful in solving this problem.

Pareto-optimal Solution

A vector $z^* \epsilon S$ is called Pareto-optimal for the problem of Eq. (24) if and only if $z \epsilon S$ and $f_i(z) \leq f_i(z^*)$, for $i = 1,2,\ldots,k$ imply that $f_i(z) = f_i(z^*)$ for $i = 1,2,\ldots,k$ where

$$S = \{ z \epsilon R^n \mid g_j(z) \leq 0, \; j = 1,2,\ldots,m_1$$
$$h_j(z) = 0, \;\; j = 1,2,\ldots,m_2 \} \tag{25}$$

Verbally, the vector z^* is Pareto-optimal if there exists no feasible vector z that would decrease some objective function without causing a simultaneous increase in at least one objective function. Usually several Pareto-optima exist for a vector optimization problem and additional information is needed to order the Pareto-optimal set. This clearly makes it possible to bring in additional considerations not included in the optimization model (besides the original objectives), thus, making the multiobjective approach a flexible technique for most design problems. Several numerical methods have been suggested for solving a vector optimization problem. Each method, in general, generates a different Pareto-optimal solution. In the most commonly used approach, known as the weighting method, a scalar objective is formulated as a weighted sum of the individual objective functions. Different Pareto-optimal solutions can be generated by varying the weights of the objective functions. A disadvantage of this method is that it is incapable of producing the whole Pareto-optimal set for certain nonconvex problems.[24] A game theory approach is presented in the present work for finding the best compromise (Pareto-optimal) solution of the multiobjective optimization problem.

Game Theory Approach

The game theory approach can be seen with reference to a two-objective, two-design variable optimization problem whose graphical representation is shown in Fig. 1. Let $f_1(z_1,z_2)$ and $f_2(z_1,z_2)$ represent two scalar objectives and z_1 and z_2 represent two scalar design variables. It is assumed that one player is associated with each objective. The first player wants to select a design variable z_1 that will minimize his payoff f_1, and similarly, the second player seeks a variable z_2 that will minimize his own payoff f_2. If f_1 and f_2 are continuous, then the contours of constant values of f_1 and f_2 appear as shown in Fig. 1. The dotted lines passing through O_1 and O_2 represent the loci of rational (minimizing) choices for the first and second player for a fixed value of z_2 and z_1, respectively. The intersection points of these two lines, if they exist, are candidates for the two-objective minimization problem assuming that the players do not cooperate with each other (non-cooperative game). In Fig. 1, the points $N_1(z_1^*,z_2^*)$ and $N_2(\underset{\sim}{z}_1^*, \underset{\sim}{z}_2^*)$ represent such points.

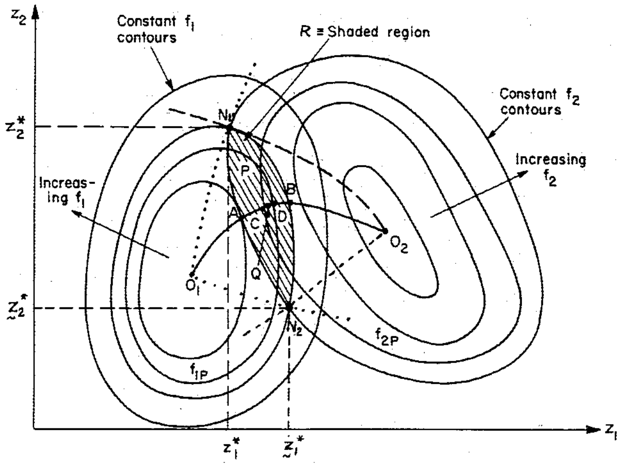

Fig. 1 Cooperative and noncooperative game solutions.

These points, known as Nash equilibrium solutions, represent stable equilibrium conditions in the sense that no player can deviate unilaterally from this point for further improvement of his own criterion.[21] These points have the characteristic that

$$f_1(z_1^*,z_2^*) \leq f_1(z_1,z_2^*) \tag{26}$$

and

$$f_2(z_1^*,z_2^*) \leq f_2(z_1^*,z_2) \tag{27}$$

where z_1 may be to the left or right of z_1^* in Eq. (26), and z_2 may lie above or below z_2^* in Eq. (27). This idea can be extended to define a Nash equilibrium solution to a k-player non-cooperative game.

In a cooperative game, the two players agree to cooperate with each other and, hence, any point in the shaded region R of Fig. 1 will provide both of them with a better solution than their respective Nash equilibrium solutions.[22] Since the region does not provide a unique solution, the concept of Pareto-optimal (non-inferior) solutions can be introduced to eliminate many solutions from the region R. It can be seen that all points in the region R can be eliminated except those on the continuous line $O_1ACQDBO_2$ that represents the loci of tangent points between the contours of f_1 and f_2. Every point on this line has the property of not being dominated by any other point in its neighborhood, i.e.,

$$f_1(Q) \leq f_1(P)$$

and

$$f_2(Q) \leq f_2(P)$$

where Q is a point lying on the line O_1O_2 (i.e., line $O_1ACQDBO_2$) and P is a neighboring point. Thus, all points of R that do not lie on the line O_1O_2 need not be considered during cooperative play. The set of all points lying on AB is known as a Pareto-optimal set and is denoted by R_p. Since R_p represents the solution set to be considered in a cooperative game, the main task in a multicriteria optimization problem will be to determine the solution set R_p.

After determining the Pareto-optimal set, one has to pick up a particular element from the set by adopting a systematic procedure. If it is possible to convert all the criteria involved in the problem to some common units, then the problem will be greatly simplified. If this is not possible, further rules of negotiation in the form of a supercriterion or bargaining model should be specified before selecting a particular element from the set R_p. In order to determine the set R_p, the following minimization problem is considered:

$$F_c(z) = c_1F_1(z) + c_2F_2(z) + \ldots + c_kF_k(z) \tag{28}$$

with

$$c_i \geq 0, \; i = 1,2,\ldots,k \tag{29}$$

and

$$\sum_{i=1}^{k} c_i = 1 \tag{30}$$

Note that the objective functions $f_i(z)$ have been normalized to $F_i(z)$ using the following procedure.

At any starting feasible design vector z_s, the objective function values are found and positive constant multipliers $m_1, m_2, \ldots, m_k$ are chosen such that

$$m_1f_1(z_2) = m_2f_2(z_s) = \ldots = m_kf_k(z_s) = M \tag{31}$$

where M is a constant. By redefining the objective functions as

$$F_i(z) = m_if_i(z), \qquad i = 1,2,\ldots,\mathrm{k} \tag{32}$$

one can notice that any design vector that minimizes the function $f_i(z)$ will also minimize the function $F_i(z)$. This scaling is done to make all the objective functions numerically equal at a particular design z_s. Hereafter, it will be assumed that the k players correspond to the k-scaled objective functions given by Eq. (32).

The Pareto-optimal set R_p is obtained by considering the various combinations of $c_1, c_2, \ldots c_k$ that satisfy Eqs. (29) and (30) and minimize $F_c(z)$ of Eq. (28) with respect to z for each combination of $c_1, c_2, \ldots c_k$. If, for a specific combination of $c_1, c_2, \ldots c_k$, $\underset{\sim}{z}$ minimizes $F_c(z)$, the set R_p can be expressed as

$$R_p = \{ \underset{\sim}{z} \text{ that minimizes } F_c(z)$$

$$\text{with } \boldsymbol{c} \text{ satisfying Eqs. (29) and (30)} \}$$

Thus, the generation of the set R_p theoretically requires the solution of several scalar minimization problems (i.e., minimizations of F_c).

Computational Procedure

The cooperative game theory approach of solving the multicriteria optimization problem can be stated as follows. The k players are assumed to correspond to the k objectives, one for each objective. While playing the game (i.e., while designing the structure), each player will try to improve his/her own conditions (i.e., to decrease the value of his/her own objective function). The players will start bargaining from their respective reference (starting) values and put a joint effort into maximizing a subjective criterion (supercriterion) formed by themselves. It is assumed that each player has analyzed his/her own criterion before starting the game to find the maximum possible benefit he/she can obtain. This will also help him/her in guaranteeing against the worst value. This analysis is necessary since each player should know the extreme conditions of his/her own and others, so that no player bargains from a reference value that is unrealistic (i.e., unacceptable to the other players). The extreme values for each player are determined as follows.

Starting from design vector z_s, each objective function $F_i(z)$, $i = 1,2,\ldots,k$ is minimized subject to the constraints stated in Eq. (24). The subroutine VMCON,[25] which is based on Powell's algorithm for nonlinear constraints that uses Lagrangian functions, is used for minimizing the objective functions $F_i(z)$. The sensitivities of the objective and constraint functions with respect to the design variables needed for the program are found numerically. Then, a matrix $[P]$ is constructed as

$$[P] = \begin{bmatrix} F_1(z_1^*) & F_2(z_1^*) & F_k(z_1^*) \\ F_1(z_2^*) & F_2(z_2^*) & F_k(z_2^*) \\ \cdot & & \\ \vdots & & \\ F_1(z_k^*) & F_2(z_k^*) & F_k(z_z^*) \end{bmatrix} \tag{33}$$

It can be seen that the diagonal elements in the matrix $[P]$ are the minima in their respective columns. Thus, the best and worst values of the objective functions can be determined as $F_i(z_i^*)$ and

$$F_{i,\max} = \max F_i(z_j^*), \quad j = 1,2,\ldots,k, \quad i = 1,2,\ldots,k \tag{34}$$

It can be seen that during the cooperative play, the ith player should not expect a value for his/her objective better than $F_i(z_i^*)$, but is guaranteed that his/her objective will never be worse than $F_{i,\max}$. Assuming that all the players start negotiation by taking their worst values as reference values, a supercriterion β can be constructed as

$$\beta = \prod_{i=1}^{k} \{ F_{i,\max} - F_i(z_c^*) \}$$

where z_c^*, a Pareto-optimal solution, minimizes a combined objective function $F_c(\boldsymbol{c}, z)$ defined by Eq. (28), subject to the constraints of Eqs. (29) and (30) and of Eq. (24).

From Eq. (35) it can be seen that all the players will be interested in maximizing the criterion β. As z_c^* is implicitly dependent on c_i, the problem now is to determine the optimum values of c_i for which β of Eq. (35) attains a maximum value. The procedure of obtaining the optimum vector $\boldsymbol{c}^* = \{c_1^*, c_2^*, \ldots, c_k^*\}^T$ begins by assuming any vector $\boldsymbol{c}$ and improving it in the subsequent iterations by moving along the steepest ascent directions of β through appropriate step lengths.

It may be noted that the construction of the supercriterion requires the solution of k single objective optimization problems. Subsequently, the supercriterion is to be optimized. The game theory approach, although involving more computational effort, gives a rational compromise solution to the multiobjective optimization problem.[26] On the other hand, if a traditional approach, such as the minimization of a linear sum of the objective functions, is used with p discrete values for each weight, then the minima of p^k single objective optimization problems are to be studied in order to have a complete trade-off study of the problem. This will be computationally more expensive than the game theory approach.

Illustrative Examples

The game theory approach is illustrated through the design of two actively controlled truss structures. The dimensions of the structure are defined in unspecified consistent units. The elastic modulus of the members is assumed to be 1.0 and the density of the structural material is taken as 0.001.

Two-Bar Truss

The two-bar truss shown in Fig. 2 is selected for its simplicity.[7] A nonstructural mass of two units is attached at node 2. The actuator and sensor are located in element 1 connecting nodes 1 and 2. The design variables (cross-sectional areas of the two bars) are bounded to lie between 10 and 2000. The passive damping ratio ζ is assumed to be zero. The first damping ratio ξ_1 is restricted to be equal to 0.1 and the second damping ratio ξ_2 is constrained to be larger than ξ_1. The initial

Table 1 Optimization results of two-bar truss

Quantity	Starting points z_o	Minimization of F_1 or F_2	Minimization of F_2 or F_4	Game theory
Design variables				
z_1	100.0	89.8788	2000.0[a]	361.8625
z_2	100.0	10.0[b]	221.6112	40.0963
Eigenvalues (open loop)				
ω_1^2	0.946	0.370	1.74	0.742
ω_2^2	1.89	1.45	6.83	2.90
Eigenvalues (closed loop)				
λ_1	$-0.331\pm0.996j$	$-0.0373\pm0.371j$	$-0.175\pm1.75j$	$-0.0746\pm0.743j$
λ_2	$-0.904\pm1.81j$	$-0.958\pm1.42j$	$-4.52\pm6.69j$	$-1.92\pm2.84j$
Damping ratios				
ξ_1	0.3153	0.1002	0.1000	0.1000
ξ_2	0.4462	0.5599	0.5599	0.5599
Objectives				
F_1	100.0	49.9394	1110.8	200.9794
F_2	100.0	127.4131	1.2139	15.7726
F_3	100.0	73.0240	255.37	98.9855
F_4	100.0	114.5173	24.277	50.0750

[a]Upper bound value. [b]Lower bound value.

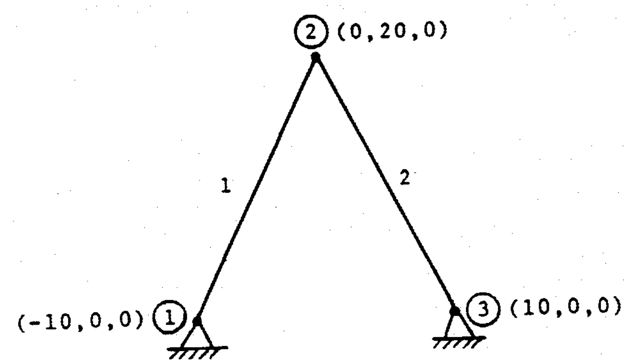

Fig. 2 Two-bar truss (sensor and actuator are located in element 1).

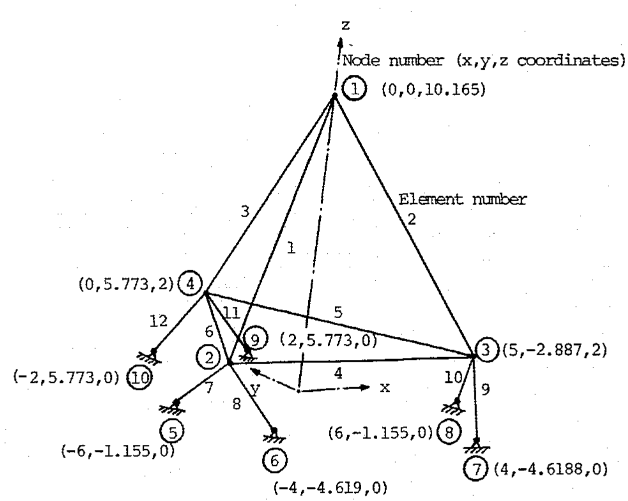

Fig. 3 Twelve-bar truss (sensors and actuators are located in elements 7–12).

disturbance vector x_0 is assumed to be same as the displacements caused by the application of a load of 1000 units at node 2 in the direction of nodes 1 to 2. It is observed that the minimization of f_1 is the same as that of f_3. Similarly, the optimum solution of f_2 is found to be the same as that of f_4. The objective functions are redefined as F_i such that $F_i = m_i f_i = 100.0$ at z_o, $i = 1,2,3,4$ with $m_1 = 22.3607$, $m_2 = 2.148\times10^{-4}$, $m_3 = 17.4167$, and $m_4 = 48.0313$. The results obtained by minimizing the individual objective functions are shown in Table 1. It may be observed that the minimization of F_1 gave a value of $F_1^* = 49.94$ with the corresponding value of $F_2 = 127.41$, while the minimization of F_2 yielded $F_2^* = 1.21$, with the corresponding $F_1 = 1110.8$. These values indicate the penalty associated with the other objective function(s) while minimizing a particular objective function.

The design problem is formulated as a game problem by considering the structural mass and controlled energy as the conflicting objective functions. The numerical solution is given in Table 1. The compromise solution may be seen to correspond to a value of $F_1 = 200.9794$ (which is much smaller than the worst possible value of 1110.8) and of $F_2 = 15.7726$ (which is much smaller than the worst possible value of 127.4131). The damping ratios ξ_1 and ξ_2 have been found to attain the same values at all the optimum solutions.

12-bar Truss (ACOSS-FOUR)

The finite-element model of ACOSS-FOUR[23] is shown in Fig. 3. The edges of this tetrahedral truss are 10 units long. The structure has 12 deg of freedom and 4 masses of 2 units each, which are attached at nodes 1–4. The actuators and sensors are located in six bipods and are assumed to coincide with each other. Thus, the matrix $[D]$ in Eq. (1) would consist of the direction cosines relating the forces in the six bipods with their components in the coordinate directions. The weighting matrix $[R]$ for this case would be of order 6×6. The passive damping parameters ζ in Eq. (4) are assumed to be zero. A unit displacement was imposed at node 2 in the x direction at $t=0$ (to define x_0).

At the starting design, the cross-sectional areas of the members are taken to be the same as those assigned by the Charles Stark Draper Laboratory in its investigation.[23] The four objective functions are redefined as $F_i = m_i f_i$ such that $F_i = 100.0$ at z_o with the values of m_i given by $m_1 = 2.2885$, $m_2 = 0.43$, $m_3 = 0.2886$, and $m_4 = 68.3045$. An equality constraint is placed on the smallest closed-loop damping ratio as $\xi_1 = \xi_1^{(o)} = 0.25$. Each of the 12 design variables (representing the cross-sectional areas of the 12 members of the truss) are restricted to lie between 10 and 2000.

Table 2 Optimization results of 12-bar truss

Quantity	Initial *design*	Minimizaton *of* F_1	Minimization of F_2	Game theory solution
F_1^a	100.0	78.2911	105.2115	78.8999
F_2	100.0	132.5619	25.5383	33.1521
F_3	100.0	48.9172	134.2457	99.6482
F_4	100.0	162.6467	23.6538	37.5417
ω_1^2 (open loop)	1.34	0.494	1.41	0.655
ω_{12}^2 (open loop)	12.9	11.8	17.1	13.2
λ_1 (closed loop)	$-0.351 \pm 1.36j$	$-0.129 \pm 0.501j$	$-0.376 \pm 1.46j$	$-0.175 \pm 0.679j$
λ_{12} (closed loop)	$-0.581 \pm 12.9j$	$-0.354 \pm 11.8j$	$-8.86 \pm 16.2j$	$-5.83 \pm 12.3j$
ξ_1	0.2503	0.2503	0.2496	0.2501
ξ_{12}	0.0451	0.0301	0.4801	0.4289

[a] $F_1 = c_i f_i$, $c_1 = 2.2885$, $c_2 = 0.4300$, $c_3 = 0.2886$, $c_4 = 68.3045$.

Table 3 Design vectors at optimum solutions

Starting vector $\bar{z}_o$	Minimization of F_1	Minimization of F_2	Game theory solution
1000	841.764	1649.545	1133.383
1000	841.864	110.259	425.194
100	13.072	220.082	19.318
100	13.072	270.997	39.231
1000	841.864	771.689	721.532
1000	841.764	782.245	690.466
100	36.013	1194.273	604.032
100	36.033	1067.297	558.441
100	10.000[a]	127.815	41.285
100	45.276	231.181	110.996
100	10.000[a]	44.908	10.000[a]
100	49.276	136.867	62.416

[a] Lower bound value.

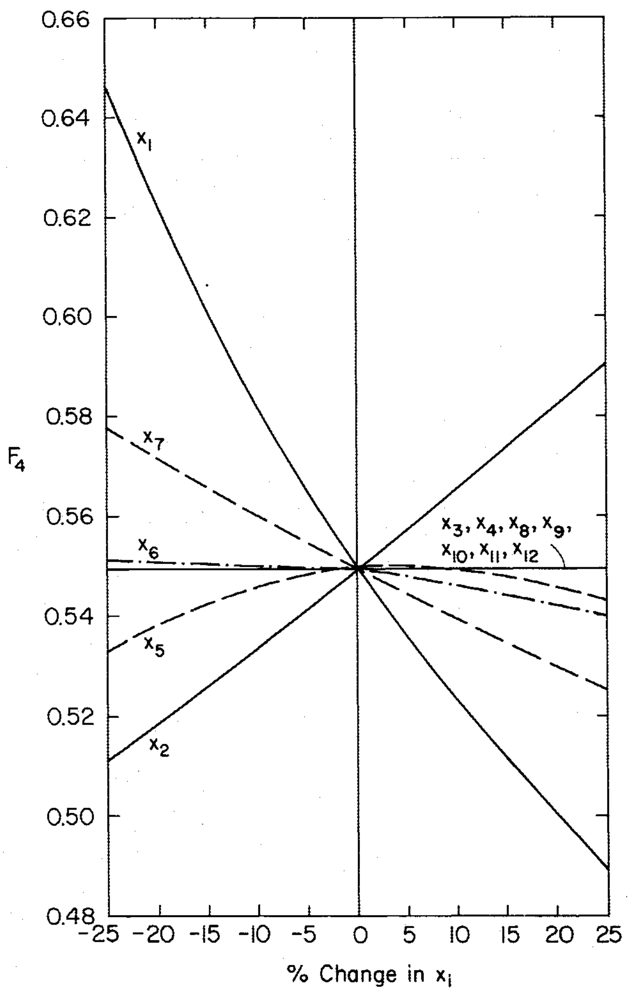

Fig. 4 Sensitivity of F_4.

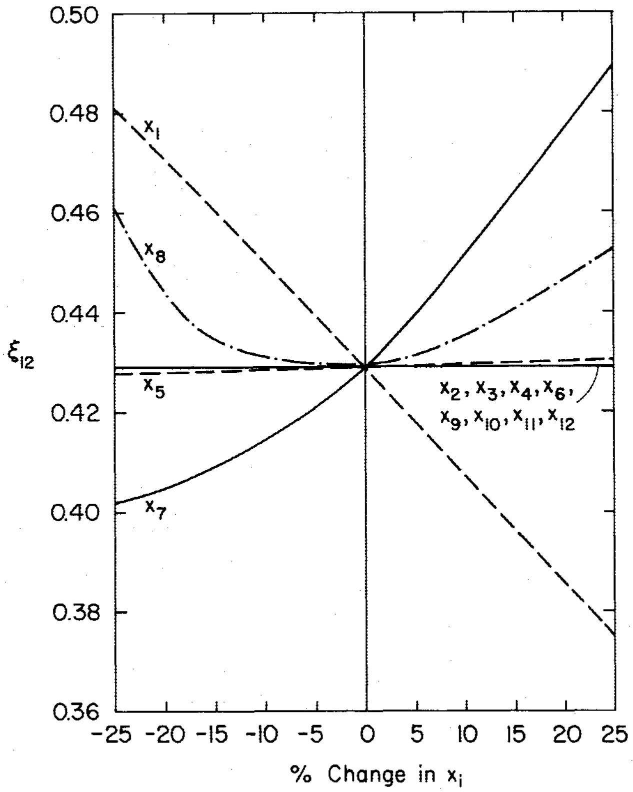

Fig. 5 Sensitivity of closed loop damping ratio ξ_{12}.

The characteristics of the starting design, along with those of the designs obtained by minimizing the individual objective functions are shown in Table 2. For this structure, also, it was observed that the minimization of $F_1(F_2)$ reduces $F_3(F_4)$. The minimum of F_1 is $F_1^* = 78.2911$ with the corresponding value of $F_2 = 132.5619$. The minimization of F_2 led to $F_2^* = 25.5383$ with the associated value of $F_1 = 105.2115$. It has been found that the minimization of weight reduced all the natural frequencies of the structure while the minimization of the controlled energy increases their values compared to the values at the starting design. The application of game theory gave the solution indicated in the last column of Table 2. This solution reduced the value of F_1 to 78.8999 (from a maximum possible value of 105.2115) and of F_2 to 33.1521 (from the worst possible value of 132.5619). This solution reduced the value of ω_1 and increased the value of ω_{12} from those at z_o. The design vectors corresponding to various solutions are shown in Table 3. The sensitivities of a typical objective function and response quantity with respect to the design variables at the game theory solution are shown in Figs. 4 and 5. These results will be useful in case the bounds on the response parameters need to be changed or the practical implementation of the optimum solution poses any difficulty.

Conclusions

1) The problem of actively controlled structures has been formulated as a multiobjective optimization problem. The concept of Pareto-optimum solution was introduced and a cooperative game theory approach was presented for finding a best Pareto-optimum solution.

2) Although only two objective problems were considered for illustration, the theory presented is general and applicable for the solution of any k-objective optimization problem.

3) It was observed, for the examples considered, that the minimization of structural weight (control energy) yielded same results as the minimization of the Frobenious norm of the control gains (effective damping response time). Additional work is needed to validate this observation.

4) The sensitivity analysis results presented in this work are expected to be useful in a) eliminating less sensitive variables from the design vector in the design of similar structures, b) rounding the theoretical optimum solution to practical available values, and c) studying the effect of changing the bounds on the response quantities.

Acknowledgments

The research work reported in this paper was conducted at the Flight Dynamics Laboratory, Wright-Patterson Air Force Base, during summer 1986. The support extended to the first author (S. S. Rao) through the USAF-UES Summer Faculty Research Program is gratefully acknowledged.

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